Wavelets – a Mathematical Microscope

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Plan for the talk

- Introduction - wavelet analysis without wavelets
- What is a wavelet?
- Why are wavelets interesting?
- Various applications
- What is a "good" wavelet?
- Outlook - aspects of harmonic analysis

Wavelets - the fear of the criminals!

How to think about wavelets

Description of how to travel from Virum to Malaga:

- Very rough description: look at a map of Europe,

  Copenhagen → Malaga

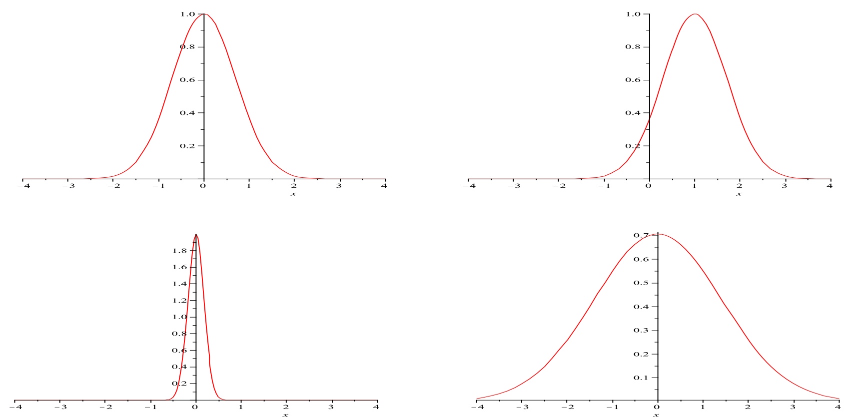
- Slightly more details: Virum → Copenhagen Airport → Malaga
- More details: Virum → Kongevejen → Klampenborgvej → Copenhagen Airport → Malaga
- Very detailed description: Virum → turn right → turn left → Kongevejen → Klampenborgvej → Copenhagen Airport → Malaga

Key feature of wavelets:

- Wavelets is a mathematical tool that can be used to describe signals or data sets;
- Wavelets allow one to zoom in on details in the given signal or data set
- It is possible to consider a very detailed representation of the entire signal;
- It is possible to consider a rough description of the given signal;
- It is possible to consider a mixed representation, giving a rough description of parts of the signal and a very detailed representation of other parts of the signal.

Definition of a wavelet

For a function ψ, let $\psi_{j,k}(x) := 2^{j/2}\psi(2^j - k)$, $j,k \in \mathbb{Z}$.



$\psi(x) = e^{-x^2}$; $\psi_{0,1}(x) = e^{-(x-1)^2}$; $\psi_{2,0}(x) = 2\psi(2^2x)$; and $\psi_{-1,0}(x) = 2^{-1/2}\psi(2^{-1}x)$.

Definition of a wavelet

Let

$$L^2(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty\}.$$

Hilbert space w.r.t. the inner product $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx$.
Definition: A wavelet is a function $\psi \in L^2(\mathbb{R})$ for which the functions $\psi_{j,k}$, $j,k \in \mathbb{Z}$, form an orthonormal basis for $L^2(\mathbb{R})$.
If ψ is a wavelet, then each function $f \in L^2(\mathbb{R})$ has a representation

$$f = \sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k},$$

where

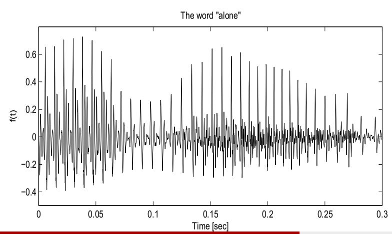
$$c_{j,k} = \langle f, \psi_{j,k}\rangle = \int_{-\infty}^{\infty} f(x)2^{j/2}\psi(2^jx - k)\, dx.$$

Application - analog to digital converter

- Playing a piece of music, the current running through the loudspeaker cable can be described by a function $f \in L^2(\mathbb{R})$.
- The representation

$$f = \sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k}$$

  converts the signal f into a sequence of numbers $\{c_{j,k}\}_{j,k\in\mathbb{Z}}$.
- All information about the music is contained in the numbers $\{c_{j,k}\}_{j,k\in\mathbb{Z}}$!
- Via the coefficients $\{c_{j,k}\}_{j,k\in\mathbb{Z}}$ we can store the signal f electronicaly (CD, DVD), or transmit the information (Internet)



Wavelets in real life

In practice:

- The wavelet representation of a signal f,

$$f = \sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k},$$

has to be replaced by

$$f \approx \sum_{(j,k)\in\mathcal{A}} c_{j,k}\psi_{j,k}$$

for an appropriate *finite* index set $\mathcal{A}$.

- It is desirable that the sets $\mathcal{A}$ associated with the relevant signals f are relatively small.

Theory versus applications:

- It is *impossible* to find a wavelet that leads to efficient representations

$$f \approx \sum_{(j,k)\in\mathcal{A}} c_{j,k}\psi_{j,k}$$

(i.e., small sets $\mathcal{A}$) simultaneously for all signals $f \in L^2(\mathbb{R})$.

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- It is often possible to find a wavelet that leads to efficient representations of all signals in a class of signals with specified properties (i.e., in a subspace of $L^2(\mathbb{R})$).
- Concrete examples of such specified signals: music files or natural images.

The first wavelet

In 1909, Alfred Haar showed that the function

$$\psi(x) = \begin{cases} 1 & \text{if } x \in [0, 1/2[\\ -1 & \text{if } x \in [1/2, 1[\\ 0 & \text{if } x \notin [0, 1[\end{cases}$$

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In 1987 Mallat and Meyer invented a general scheme for construction of a wavelet.
Necessary tool: Operators on $L^2(\mathbb{R})$.
Operators on $L^2(\mathbb{R})$: Translation,

$$\text{Translation: } (T_k f)(x) := f(x-k), \quad \text{Dilation: } (Df)(x) := 2^{1/2} f(2x).$$

Observe that

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) = D^j T_k \psi(x).$$

Multiresolution analysis I

Definition: A multiresolution analysis for $L^2(\mathbb{R})$ consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ and a function $\phi \in V_0$, such that the following conditions hold:

(i) The spaces V_j are *nested, i.e.,*

$$\cdots V_{-1} \subset V_0 \subset V_1 \cdots.$$

(ii) $\overline{\cup V_j} = L^2(\mathbb{R})$ and $\cap V_j = \{0\}$.

(iii) For all $j \in \mathbb{Z}$, $V_{j+1} = DV_j$.

(iv) $f \in V_0 \Rightarrow T_k f \in V_0, \ \forall k \in \mathbb{Z}$.

(v) $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0.

Multiresolution analysis II - key ingredients:

Consequence of the MRA-setup: the spaces V_j and the function ϕ are related by

$$V_j = \overline{span}\{D^j T_k \phi\}_{k\in\mathbb{Z}}.$$

Fourier transformation: $\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\gamma}\, dx$.

Lemma: Assume that the function $\phi \in L^2(\mathbb{R})$ generates a multiresolution analysis. Then there exists a 1-periodic function $H_0 \in L^2(0,1)$ such that

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma).$$

This is the *scaling equation.*

Multiresolution analysis III - the wavelet

Theorem: Assume that $\phi \in L^2(\mathbb{R})$ generates a multiresolution analysis, and let $H_0 \in L^2(0,1)$ be a 1-periodic function satisfying the scaling equation. Define the 1-periodic function H_1 by

$$H_1(\gamma) := \overline{H_0(\gamma + \frac{1}{2})} e^{-2\pi i\gamma}.$$

Then, the function ψ defined via

$$\hat{\psi}(2\gamma) = H_1(\gamma)\hat{\phi}(\gamma) \tag{1}$$

is a wavelet. Explicitly, if

$$H_1(\gamma) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi i k\gamma},$$

then

$$\psi(x) = 2\sum_{k\in\mathbb{Z}} c_k \phi(2x+k).$$

Multiresolution analysis IV - the consequences

If ψ is a wavelet, then each $f \in L^2(\mathbb{R})$ has a representation

$$f = \sum_{j,k \in \mathbb{Z}} c_{j,k} \psi_{j,k},$$

where

$$c_{j,k} = \langle f, \psi_{j,k} \rangle = \int_{-\infty}^{\infty} f(x) 2^{j/2} \psi(2^j x - k)\, dx.$$

Multiresolution analysis V - the key result

Alternatively,

$$\{T_k\phi\}_{k\in\mathbb{Z}} \cup \{\psi_{j,k}\}_{j\geq 1, k\in\mathbb{Z}}$$

forms an orthonormal basis for $L^2(\mathbb{R})$. Thus, for any $f \in L^2(\mathbb{R})$,

$$f = \sum_{k\in\mathbb{Z}} \langle f, T_k\phi\rangle T_k\phi + \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}.$$

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Wavelets are like a microscope: keeping more and more terms of the infinite sum $\sum_{j=1}^{\infty}$ will give a more and more detailed representation of the signal.

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Wavelet analysis in practice

The exact representation

$$f = \sum_{k\in\mathbb{Z}} \langle f, T_k\phi\rangle T_k\phi + \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}$$

$$= \sum_{k\in\mathbb{Z}} \langle f, T_k\phi\rangle T_k\phi + \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle 2^{j/2}\psi(2^j x - k)$$

must be replaced by a finite sum.

- Assume that ϕ and ψ have compact support, i.e., vanish outside a certain bounded interval.
- Let $d_{j,k} := 2^{j/2}\langle f, \psi_{j,k}\rangle$.

The Haar wavelet

Example: For the *Haar wavelet*

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \le x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

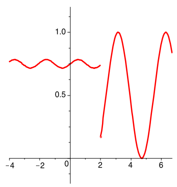
it can be shown that

$$d_{j,k} := 2^{j/2}\langle f, \psi_{j,k}\rangle = \frac{1}{2}\Big(\text{average of } f \text{ over } 2^{-j}[k, k+1/2[\; - \text{average of } f \text{ over } 2^{-j}[k+1/2, k+1[\Big).$$

Ex: $d_{1,1} = \frac{1}{2}(\sharp[\frac{1}{2}, \frac{3}{4}] - \sharp[\frac{3}{4}, 1])$, $d_{2,3} = \frac{1}{2}(\sharp[\frac{3}{4}, \frac{7}{8}] - \sharp[\frac{7}{8}, 1])$,
$d_{3,6} = \frac{1}{2}(\sharp[\frac{12}{16}, \frac{13}{16}] - \sharp[\frac{13}{16}, \frac{14}{16}])$,

The Haar wavelet

$$d_{j,k} = \frac{1}{2}\Big(\text{av. of } f \text{ over } 2^{-j}[k, k+1/2[\; - \text{av. of } f \text{ over } 2^{-j}[k+1/2, k+1[\Big).$$



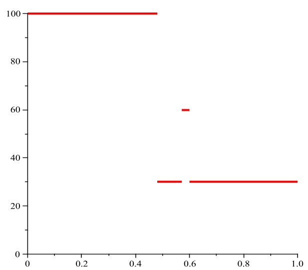
- For j, k corresponding to intervals where f is almost constant:

$$d_{j,k} \approx 0$$

- Effective compression by putting $d_{j,k} = 0$.
- For j, k corresponding to intervals where f jumps: $d_{j,k}$ is large!

Digital black-white images

- A digital consists of a number of pixels, e.g.,256 × 256 pixels.
- To each pixel a graytone is associated, e.g., on a scale from 0 to til 100:
  0: completely white   100: completely black



Pixel values in "first row."

- Wavelets led to an efficient representation!

An alternative description

We consider a small example with eight numbers instead of 256 pixel values.

- Order the eight numbers

56	40	8	24	48	48	40	16

  as four pairs consisting of two numbers.
- Replace each pair (a, b) by $(c, d) = (\frac{a+b}{2}, a - \frac{a+b}{2})$:

Averages:

48	16	48	28

Differences:

8	-8	0	12

We repeat the process:

Digital signal processing

Given numbers:

56	40	8	24	48	48	40	16

Replace each pair (a, b) by $(c, d) = (\frac{a+b}{2}, a - \frac{a+b}{2})$:

48	16	48	28	8	-8	0	12

We apply the method again, but only on the 4 averages:

32	38	16	10	8	-8	0	12

We apply the method again, but only on the 2 averages:

35	-3	16	10	8	-8	0	12

Thresholding: throw away all "small numbers". Gives an efficient compression

The story of FBI

- Before 1995, had fingerprints for 30 million people, stored on paper charts.
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Solution: Compression!

The story about FBI

The story about FBI

 

Original fingerprint, and compressed version.
A fingerprint can be stored using 1 Mb instead of 13 Mb - without loss of quality!

The story about FBI

 

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Does it work?

The story about FBI

Original fingerprint, and compressed version.
A fingerprint can be stored using 1 Mb instead of 13 Mb - without loss of quality!
Does it work?
The first year after introducing the method FBI solved 800 cases that were thought to be unsolvable (50 cases involving a person being killed).

Other applications

- Recognition of the Iris;
- Wavelets are part of JPEG2000 (international standard for image compression);
- Video transmission;
- Compression of music (MP3-players);
- Noise reduction;

Applications

- **Noise reduction:** In the representation

$$f = \sum_{k\in\mathbb{Z}} \langle f, T_k\phi\rangle T_k\phi + \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k},$$

  large values of j correspond to high frequencies.
- Filtering of high frequencies is done by replacing this by

$$f = \sum_{k\in\mathbb{Z}} \langle f, T_k\phi\rangle T_k\phi + \sum_{j=1}^{J}\sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k},$$

  for a sufficiently high value of J.

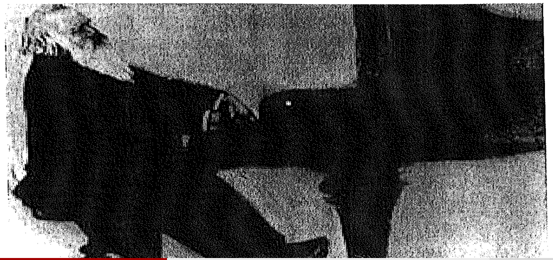
Johannes Brahms: Composer and pianist

Recorded in 1889 *Ungarischen Tanzes Nr.1*

- Noisy recording, impossible to hear that a piano is playing
- Wavelet methods: removal of the noise
- The cleaned recording reveals how Brahms played his own music
- Surprise: What Brahms played was quite different from what he wrote in the score!

The score:

What Brahms played:



Other wavelets - or is the Haar wavelet good enough?

Recall: given a wavelet ψ arising from a multiresolution analysis generated by the function ϕ, we have

$$\begin{aligned} f &= \sum_{k\in\mathbb{Z}} \langle f, T_k\phi\rangle T_k\phi + \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k} \\ &= \sum_{k\in\mathbb{Z}} \langle f, T_k\phi\rangle T_k\phi + \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} 2^{j/2}\langle f, \psi_{j,k}\rangle \psi(2^j x - k). \end{aligned}$$

For the Haar wavelet,

$$\begin{aligned} d_{j,k} := 2^{j/2}\langle f, \psi_{j,k}\rangle &= \frac{1}{2}\Big(\text{average of } f \text{ over } 2^{-j}[k, k+1/2[\\ &\qquad - \text{average of } f \text{ over } 2^{-j}[k+1/2, k+1[\Big). \end{aligned}$$

Thus,

$$|\langle f, \psi_{j,k}\rangle| \le C\,2^{-j/2}.$$

Vanishing moments

Recall: For the Haar wavelet, $|\langle f, \psi_{j,k}\rangle| \le C\,2^{-j/2}$.
Definition: Let $N \in \mathbb{N}$. A function ψ has N vanishing moments if

$$\int_{-\infty}^{\infty} x^{\ell}\psi(x)\,dx = 0 \text{ for } \ell = 0, 1, \ldots, N-1.$$

For the Haar wavelet: $N = 1$.

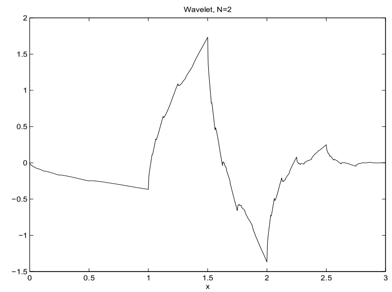
Theorem: Assume that the function $\psi \in L^2(\mathbb{R})$ is compactly supported and has N vanishing moments. Then, for any $N-$times differentiable function $f \in L^2(\mathbb{R})$ for which the Nth derivative $f^{(N)}$ is bounded, there exists a constant $C > 0$ such that

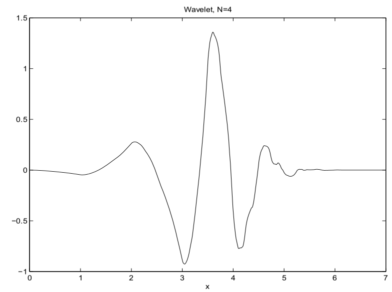
$$|\langle f, \psi_{j,k}\rangle| \le C\,2^{-jN}2^{-j/2}, \ \forall j, k \in \mathbb{Z}.$$

Daubechies wavelets

Daubechies' wavelets - family parametrized by $N \in \mathbb{N}$:
Given $N \in \mathbb{N}$, the Nth Daubechie's wavelet ψ_N has

- N vanishing moments;
- supp $\psi_N = [0, 2N-1]$.



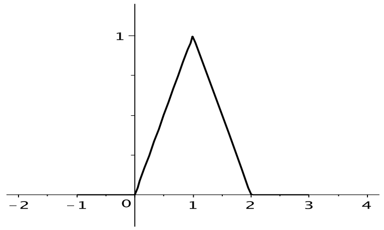


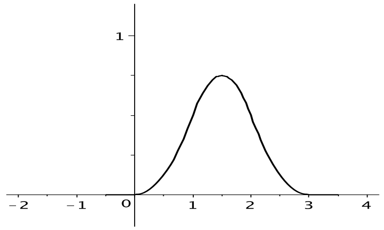
Variations

- Wave packets
- Biorthogonal wavelets
- Ridgelets
- Curvelets
- Contourlets
- Coiflets
- Framelets (wavelet frames)

Wavelet frames

B-splines: $B_1 := \chi_{[0,1]}, \ B_{N+1}(x) := B_N * B_1(x) = \int_0^1 B_N(x-t)\,dt.$





For any B-spline B_N there exist two functions

$$\psi_1(x) = \sum c_{k,1}B_N(2x+k), \quad \psi_2(x) = \sum c_{k,2}B_N(2x+k)$$

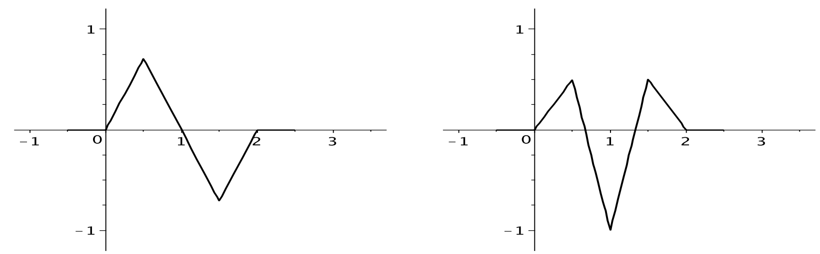
such that

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, (\psi_1)_{j,k}\rangle(\psi_1)_{j,k} + \sum_{j,k\in\mathbb{Z}} \langle f, (\psi_2)_{j,k}\rangle(\psi_2)_{j,k}.$$

Wavelets and Mechanics

- A spline wavelet finite-element method in structural mechanics (Han, Ren, Huang)
- Online Identification of Linear Time-varying Stiffness of Structural Systems by Wavelet Analysis (Basu, Nagarajaiah, Chakraborty)
- Hinge-free topology optimization with embedded translation-invariant differentiable wavelet shrinkage. (Bendsøe, Kim, Sigmund,Yoon)

Wavelet frames based on two generators



An alternative to wavelets

Goal: Expansions of complicated functions/signals/sequences f

$$f = \sum c_k f_k$$

in terms of convenient building blocks f_k.

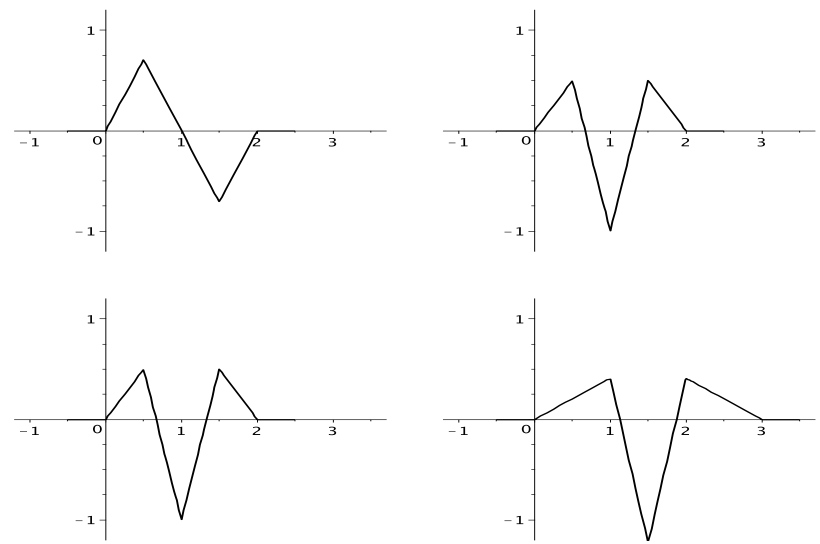
- One choice of f_k - a wavelet system
- Fourier expansion of 1-periodic function:

$$f = \sum_{m \in \mathbb{Z}} c_m e^{2\pi i m x}, \quad c_m = \int_0^1 f(x) e^{-2\pi i m x}\, dx.$$

- Fourier-like expansions of functions $f \in L^2(\mathbb{R})$:

$$f = \sum_{m,n \in \mathbb{Z}} \langle f, e^{2\pi i m x} \chi_{[0,1]}(x-n) \rangle e^{2\pi i m x} \chi_{[0,1]}(x-n).$$

Wavelet frames based on two generators



Gabor systems

Given a function $g \in L^2(\mathbb{R})$ and two parameters $a, b > 0$, the associated Gabor system is

$$f_k \sim E_{mb}T_{na}g(x) = e^{2\pi imbx}g(x - na).$$

- Applications: Digital audio broadcasting, communication technology.
- We search for a Gabor expansion,

$$f = \sum_{m,n\in\mathbb{Z}} c_{m,n}E_{mb}T_{na}g, \; f \in L^2(\mathbb{R}).$$

- $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ can be an ONB for $L^2(\mathbb{R})$: take

$$g = \chi_{[0,1]}, a = b = 1.$$

Gabor systems in time-frequency analysis

Bad news for applications in time-frequency analysis:

Theorem: *Assume that g is continuous and has compact support. Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ can not be a basis for $L^2(\mathbb{R})$.*

No nice ONB exists!

Frames in Hilbert spaces

Definition: *A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in a Hilbert space H is a frame for H if there exist constants $A, B > 0$ such that*

$$A\,||f||^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\,||f||^2, \;\; \forall f \in H.$$

Theorem: Assume that $\{f_k\}_{k=1}^{\infty}$ is a frame. Then there exists another frame $\{g_k\}_{k=1}^{\infty}$ for which

$$f = \sum_{k=1}^{\infty} \langle f, g_k\rangle f_k, \; \forall f \in H.$$

Desirable properties if $H = L^2(\mathbb{R})$:

- f_k, g_k are explicitly given functions with compact support;
- $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ have Gabor or wavelet structure.
- $\widehat{f_k}$ and $\widehat{g_k}$ decay fast.

Explicit construction of dual pairs of Gabor frames

Theorem: ([C., Kim, 2007]) *Let $N \in \mathbb{N}$. Let $g \in L^2(\mathbb{R})$ be a real-valued bounded function for which*

- *supp* $g \subseteq [0, N]$,
- $\sum_{n\in\mathbb{Z}} g(x - n) = 1$.

Let $b \in]0, \frac{1}{2N-1}]$ and define the function h by $h(x) = b\sum_{n=-N+1}^{N-1} g(x + n)$. Then $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$.

So

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_n h\rangle E_{mb}T_n g, \; \forall f \in L^2(\mathbb{R}).$$

For $g = B_N$:

- The functions B_N and the dual are splines;
- B_N and the dual have compact support, i.e., perfect time–localization;
- By choosing N sufficiently large, polynomial decay of $\widehat{B_N}$ and the dual of any desired order can be obtained.